A Contribution to the Theoretical Prediction of Life-time in Glass Structures

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1 Basic Concepts about the Mechanical Behaviour of Glass

Every glass surface, although apparently intact, is unavoidably affected by microscopic randomly distributed cracks. When the glass element is subjected to mechanical stresses, high stress concentrations occur at the tip of the micro cracks which can not be plastically redistributed because of the amorphous crystalline structure of the material, lacking in preferential plastic-flow plans. This peculiar feature causes the typical brittle fractures that characterize this material. The fracture resistance of damaged elements can be analytically described by the principles of *Linear Elastic Fracture Mechanics*. For this reason Irwin [1] introduced the *Stress Intensity Factor (K)*, in order to describe the behaviour of brittle materials damaged by a single flaw placed perpendicularly to the stress direction (opening mode I):

$$K(t) = \sigma(t) Y \sqrt{\pi a(t)} \qquad (1\text{-}0)$$

Failure occurs when the propagation of the crack becomes unstable, that happens when:

$$K(t) \geq K_{IC} \qquad (1\text{-}0)$$

K_{IC} represents the *Critical Stress Intensity Factor* which depends only on the kind of material and can be usually considered technically constant because of its low statistical spread. Substituting (1-2) in (1-1) easily allows to obtain the $\boldsymbol{a_{cr}}$ e $\boldsymbol{\sigma_{cr}}$ analytic

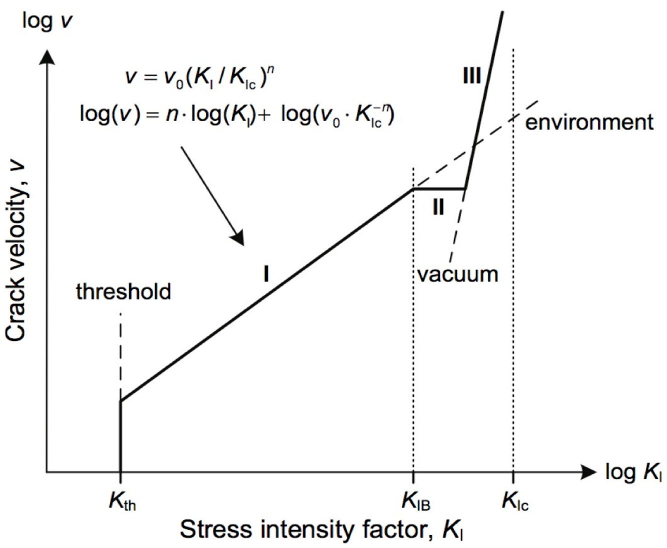

Fig. 1-1 – υ versus K_I

expressions, respectively representing the crack depth and the stress intensity able to induce unstable crack propagation. This pair of values identifies the so-called "*inert strength*". The graph of Fig. 1-1 shows, according to the K-factor, the flaw propagation velocity of a glass element subjected to constant stress during time and immersed in a humid environment. Although in section I the K value is much lower than K_{IC}, a slow sub-critical growth of flaws depth occurs on the glass surface which gradually reduces the inert tensile glass strength over time. This phenomenon is known as s*tatic fatigue* and plays one of the main roles in theoretically determining the ultimate strength of glass structures. As shown in Fig.1-1, the *v*-K relation is represented by a constant slope curve on a bi-logarithmic plot and it can be analytically described by the following differential equation:

$$\partial a/\partial t = v_0 \left(K(t)/K_{IC} \right)^n \tag{1-0}$$

with *n* being the curve's slope in section I and v_0 the propagation velocity when $K = K_{IC}$.

2 Haldimann's Probabilistic Method: *the Lifetime Prediction Model*

At present, the most advanced prediction model of strength in glass element seems to be the *Lifetime Prediction Model* formulated by M.Haldimann in [2]. He demonstrates that the crack opening mode I mainly affects failure probability (P_f) finally finding the following general expression of P_f that describes the life-time of a glass element of whatever shape, submitted to sub-critical cracks' growth and to a generic time and space variable stress-history :

$$P_f(t) = 1 - \exp\left\{ -\frac{2}{A_0 \pi} \int_A \int_{\varphi=0}^{\pi/2} \left[\max_{\tau\in[0,t]} \left\{ \left(\left(\frac{\sigma(\tau, \vec{r}, \varphi)}{\vartheta_0} \right)^{n-2} + \frac{1}{U \vartheta_0^{n-2}} \int_0^{\tau} \sigma^n(\tilde{\tau}, \vec{r}, \varphi) \, d\tilde{\tau} \right)^{\frac{1}{n-2}} \right\} \right]^{m_0} dA \, d\varphi \right\} \tag{2-0}$$

Where:

- θ_0 and m_0 are statistical parameters, related to the Weibull distribution, that describe the damage rate of the surface. They can be determined by experiments and are material intrinsic properties, not dependant on the type of laboratory tests.
- $U = \frac{2K_{IC}^2}{\left[(n-2) v_0 Y^2 \pi\right]}$ is an expression related to specific parameters of the material, usually characterized by constant values, defined by *Linear Elastic Fracture Mechanics* and by the *Static Fatigue* differential equation.

The (2-1) is therefore related, by means of a probabilistic approach, to the parameters θ_0 and m_0 which are characterized by a clear physical meaning [2 -3]. Restrictive assumptions are not stated about element shape, load or stress time-history and space variability, constraints and damaging surface condition. The only conceptual limitation of the Lifetime Prediction Model is that loads are assumed to be deterministic variables. The analytical complexity of the expression (2-1) make it not suited for current engineering oriented design activities. For this reason, Haldimann himself suggested a simplified version of it by introducing some conservative assumptions [2]. After some manipulation, expression (2-1) finally reduced to the following simplified expression of the failure probability (the meaning of $\bar{\sigma}$ is described in paragraph 4 by expressions (4-1) and (4-2):

$$P_f = 1 - \exp(-k \cdot \bar{\sigma}^{m'}) \tag{2-0}$$

Where $k = t_0 / U \cdot \vartheta_0^{n-2}$ and $m' = n \cdot m_0 / n - 2$. Then, once selected a given P_f for the glass tensile strength, a *failure criterion* can be written in the following form:

$$\text{Stress} \;=\; \boxed{\bar{\sigma} \le \left[-\ln(1-P_f)\right]^{1/m'} \cdot \left(\frac{t_0}{U \cdot \theta_0^{n-2}}\right)^{\frac{-1}{n}}} \;=\; \text{Strength}_{\,probabilistic} \tag{2-0}$$

3 Deterministic Model: Single Crack Life Time

The (1-1) and (1-3) describe glass mechanical behaviour during time of an ideal perfect element only damaged with a single flaw, referring respectively to the Linear Elastic Fracture Mechanics and to the static fatigue phenomenon. Let us suppose that the general surface stress time-history is uniform over the surface and acting for *T*-seconds. If the initial crack size is also known, substituting (1-1) in (1-3) leads to the following integral-differential equation with separable variable:

$$\int_0^t \sigma^n(\tau)\left[v_0\left(\frac{Y\sqrt{\pi}}{K_{IC}}\right)^n\right] d\tau = \int_{a_i}^{a_f} a^{-\frac{n}{2}}\, da \tag{3-0}$$

Taking into account (1-2), formula (3-1) can be rearranged into (3-4) which states that a brittle unstable crack propagation does not occur if the following condition is satisfied:

$$\int_0^T \sigma^n(t)\,dt \le \frac{2 \cdot K_{IC}^n}{(n-2)v_0(Y\sqrt{\pi})^n a_i^{\frac{n-2}{2}}}\left[1-\left(a_i \Big/ \left(\frac{K_{IC}}{\sigma(t)Y\sqrt{\pi}}\right)^2\right)^{\frac{n-2}{2}}\right] \tag{3-0}$$

Finally, assuming the conservatory hypothesis that $a_i <<< a_{cr}$ (as demonstrated by Haldimann [2] for common load application durations), the quantity in square brackets approaches 1 and therefore it is possible to separate the variables obtaining the following inequality where the damage caused by external loading is compared with the maximum damage that glass can withstand:

$$D_{amage_{Soll.}}(\sigma, t) \le D_{amage_{Max.Tolerable}}(K_{IC}, n, v_0, Y, a_i) \tag{3-0}$$

If we rewrite (3-3) in an explicit way we obtain:

$$\int_0^T \sigma^n(t)\,dt \le \frac{2K_{IC}^n}{(n-2)v_0(Y\sqrt{\pi})^n}\, a_i^{\frac{2-n}{2}} \tag{3-0}$$

where the first part of (3-4) is the well known *Brown's Integral*. Therefore, if we know a generic stress time-history lasting T-seconds, by arbitrarily choosing the value of t_0 reference time, it is possible to calculate by Brown's Integral the *equivalent constant tensile stress* (see section 4 and 5) that induces onto the glass surface, during t_0, the same damage as the real stress history variable over the time T:

$$\sigma_{t_0} = \left[\frac{1}{t_0} \int_0^T \sigma^n(t)\, dt \right]^{\frac{1}{n}} \tag{3-0}$$

With position (3-5) the *failure criterion* (3-4) can be written in the following form, where the deterministic strenght is a function only of the initial crack a_i (because the other parameters are characterized by low statistical spread [4]):

$$\text{Stress} \quad = \quad \boxed{\sigma_{t_0} \le \left[\frac{1}{t_0} \frac{2K_{IC}^n}{(n-2)\cdot v_0 \cdot (Y\sqrt{\pi})^n} a_i^{\frac{2-n}{2}} \right]^{\frac{1}{n}}} \quad = \quad \text{Strength}_{\ deterministic} \tag{3-0}$$

4 The Design Crack Method (DCM)

4.1 Basic idea

In paragraph 3 we have briefly recalled the deterministic model of the mechanical behaviour of an ideal perfect glass plate, only containing a single flaw with a known initial depth a_i, submitted to a uniform tensile stress $\sigma(t)$ generically variable during a time T.

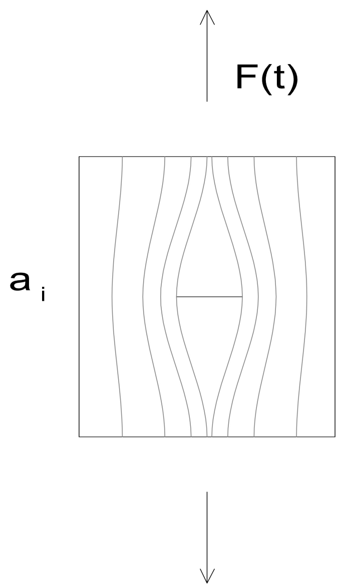

Fig. 5-1 – Basic Idea

The two main advantages of this model consist in its analytical simplicity and in the reliability of the solution, but on the other hand it does not take into account the randomness of the main parameters affecting the problem of glass tensile strength during time.

In order to overcome this problem, it was thought to search the analytical expression of a single *"**Design Crack**"* having such a depth $\boldsymbol{a_{i,d}}$ able to induce the same damaging rate of the real glass element subject to a random distribution of cracks over its surface. It was also decided to pursue this goal analytically, without using any empirical coefficient or assumption.

In the following expression, $\sigma_{t0}(x,y)$ represents the constant tensile stress of duration t_0 equivalent to the real stress history $\sigma(t,x,y)$ generally variable during time T, while $\bar{\sigma}$ represents the uniformly distributed and constant over time t_0 tensile stress acting across the element of area A, equivalent to (in terms of damage) any generic $\sigma(t,x,y)$ (see also [3]). Analytically :

$$\sigma_{t_0}(x,y) = \left[\frac{1}{t_0} \int_0^T \sigma^n(t,x,y)\, dt \right]^{\frac{1}{n}} \approx \left[\frac{1}{t_0} \sum_{i=1}^{k} \sigma(x,y)_i^n \cdot \Delta t_i \right]^{\frac{1}{n}} \tag{4-0}$$

$$\bar{\sigma} = \left[\frac{1}{A_0} \int_A \sigma_{t_0}^{m'}(x,y)\, dA \right]^{\frac{1}{m'}} \approx \left[\frac{1}{A_0} \sum_{j=1}^{q} \sigma_{t_0,j}^{m'} \cdot \Delta A_j \right]^{\frac{1}{m'}} \tag{4-0}$$

Obviously the integrals are extended only to decompressed areas, referring to surface stress field net of compression residual stresses induced by tempering processes and external pre-stressing.

4.2 Analytical Formulation of the Design Crack Method

The problem is analytically stated by equating the material strength expressed in a deterministic way by the second term of equation (3-6) to the probabilistic strength expressed by equation (2-3):

$$R_{\det}(v_0, n, Y, K_{IC}, t_0, a_{i,d}) \quad = \quad R_{probabilistic}(P_f, \vartheta_0, m_0, n, v_0, Y, K_{IC}) \qquad (4\text{-}0)$$

It can be seen that the time-term t_0 can be eliminated and therefore, after some re-arranging, we achieve the final time-independent expression of the ***Design Crack*** $a_{i,d}$,

$$a_{i,d} = \left[\left(\frac{K_{IC}}{Y \vartheta_0} \right)^2 \frac{(-\ln(1-P_f))^{-2/m_0}}{\pi} \right] = a_{i,d}\,(P_f,\ \theta_0,\ m_0) \qquad (4\text{-}0)$$

Combining (4-4) with (3-3), we define at first the failure criterion in terms of damage:

$$D_{amage_{Soll.}}(\sigma, t) \quad \leq \quad D_{amage_{Max.Tolerable}}(a_i) \qquad (4\text{-}0)$$

After some re-arrangement, the preceding *failure criterion* related to a glass element submitted to any stress time-history over a time interval can be written in more common terms of tensile stresses:

$$\text{Stress} = \quad \bar{\sigma} \leq \left[\frac{1}{t_0} \frac{2 K_{IC}^{n}}{(n-2) \cdot v_0 \cdot (Y \sqrt{\pi})^n} a_{i,d}(P_f, \vartheta_0, m_0)^{\frac{2-n}{2}} \right]^{\frac{1}{n}} = \text{Strength}_{\,Semi\text{-}Probabilistic} \qquad (4\text{-}0)$$

The final glass strength equation illustrated by equation (4-6) is thus reached by using the probabilistic parameters $a_{i,d}$ (P_f , θ_0 , m_0) defined by (4-4) together with the deterministic strength criterion described by (3-6). It can be affirmed that the present criterion belongs to the so-called **semi-probabilistic** safety verification processes (level 1), where t_0 is arbitrarily chosen and K_{IC}, n, Y, thanks to their low statistical spread [4], can be technically characterized by constant values. As demonstrated in [3], by numerical solution stability test, the variations of other parameters do not influence significantly the solution of equation (4-6) which exhibits a stable behaviour. In Design Crack Method, the failure probability is directly contained in (4-4) and the aspects linked to glass material characteristics come into play by the parameters θ_0 e m_0 obtained by Haldimann in [2], statistically analyzing a large number of failure tests by L.P.M changing some main factors such as geometry, environmental conditions, load shape and load increasing velocity. For this reason Haldiman's values θ_0 and m_0 are characterized by the highest reliability level and they will be adopted for the following numerical applications.

5 Numerical Applications

5.1 Existing Methods

In [5,3] are presented the results of a numerical comparison performed, both for short duration (60s) and long duration (50 years) load time-history, on the most commonly used criteria of glass strength as *Life Duration Theory, Crack Growth Model, Glass Failure Prediction Model, Modified Crack*

Growth Model, Sedlacek's model. The choice of two very different loading times was necessary since glass is very sensitive to the so-called *Static Fatigue* effect. Most of existing methods, as L.D.T and C.G.M., show a good agreement with reference values for short time-history but are not able to describe the glass strength behaviour subjected to long duration stress field. Numerical results show also that the G.F.P.M does not provide safety values for either long or short time loading. In spite of that, this method is still adopted by some national standards like the American ASTM E 1300 and the Canadian CAN 12-20. The Sedlacek's calculation model gives stress and P_f values similar to the reference ones. The assessment process is performed by transforming the real service condition of a generic glass element into an equivalent standard laboratory test *[6,7]* by means of a set of coefficients whose knowledge and reliability is implicitly assumed [8].

5.2 Design Crack Method

Table 5-1 : D.C.M. - Numerical Applications – [5,3]

n°	**Num.Applicat.**	**Design Crack Method**	**Reference Values**
ũ	σ_{max} – **60 s**	21.76 *MPa*	20.3 *Mpa*
ĩ2	σ_{max} – **50 years**	8.27 *MPa*	8.12 *Mpa*
2. 1	P_f - ***60 s***	4.122 x 10^{-3}	8 x 10^{-3}
2. 2	P_f - **50 years**	6.734 x 10^{-3}	8 x 10^{-3}
3	$\sigma_{bB,A0}$	42.24 *MPa*	45 *MPa*
4	σ_{max} – ***10^{-10} ≈ 0 s***	34.68 *MPa*	σ_{inerth} = 34.68 *MPa*

Under the same service conditions of restrains, geometry and environment used in paragraph 5-1, some simple numerical applications of the new D.C.M. have been carried out with the results summarized in Table 6-2. We also developed a numerical simulation of the standard double-ring test in application n°3 while in application n°4 the numerical convergence of the method to the inert strength a_{cr} e σ_{cr} was controlled, for an extremely high load application rate, by numerically solving expression (3-2) instead of (3-4).

5.3 Elementary numerical examples

Each numerical value assumed in these examples has been found in technical literature, and can be improved or changed in order to reach better method calibration, achieving more accurate and experimentally validated results. Therefore the following numerical examples were developed only to provide an explanation as clear and simple as possible of the Design Crack Method

5.3.1 Constant uniform tensile stressed glass plate

Let's suppose (action side) a constant uniform tensile stress about 8 Mpa of 1 hour equi-damaging duration, net of compressive tempering stress. As we have seen before, fixing the maximium value of P_f linked to the minimum safety level that you want to guarantee (fixed by Eurocodes), we easily obtain the design crack value $a_{i,d} = 0,189\,mm$. Now choosing the reference duration of strength (for example 1 hour) we obtain the one-hour-resistance:

$$\bar{\sigma}_{max,t_0} = \left[\frac{1}{3600s} \cdot \left(8,21 \cdot 10^{-7}\right) \cdot a_{i,d}^{-7} \right]^{\frac{1}{16}} \qquad \bar{\sigma}_{max,t_0} = 10.604 MPa$$

This value does not represent the failure tensile stress. Assuming this value for glass design means to be as far from failure as fixed P_f values gaurantees. Therefore the safety semi-probabilistic inequality will be:

$$\frac{\bar{\sigma}_{t_0}}{\bar{\sigma}_{max,t_0}} = \frac{8 MPa}{10,604 MPa} = 0,754 \leq 1$$

5.3.2 Constant uniform tensile stressed glass plate – different reference strength duration

If we want to investigate different reference strength of duration t_1, according to (3-7), we have just to multiply by:

$$\cdot \left(t_0 / t_1\right)^{1/16}$$

5.3.3 More than one time-history load of different duration

In case of load combination on glass structures we must talk about cumulative damage instead of instant load combination, because of the presence of Static Fatigue. Therefore for instance in the case of three different load time durations, choosing an arbitrary duration t_0, we achieve by (4-1) the equivalent stress:

$$\bar{\sigma}_{t_0} = \left[\frac{1}{t_0} \left(\sigma_1^{16} \cdot t_1 + \sigma_2^{16} \cdot t_2 + \sigma_2^{16} \cdot t_2 \right) \right]^{\frac{1}{16}}$$

Making a numerical example, in the case of constant stress σ_1 = 1,0 MPa of duration t_1 = 50 years, costant stress σ_2 = 3,0 MPa of duration t_2 = 1 year and constant stress σ_3 = 10,0 MPa of duration $t_{3\,=}$ 10 minutes, arbitrary fixing the reference duration, t_0 = 1 hour, we obtain the equidamaging costant stress:

$$\sigma_{t0} = 8,941\ MPa \quad \textit{of duration } t_0 = 1\ hour$$

Obviously a generic time history can be subdivided in tecnically constant intervals.

5.3.4 Non uniform tensile stress field

Dismissing any allowable-stress approach, as defined by Eurocode guideline, defining A_0 as the surface of the reference glass speciments used for the laboratory determination of statistical surface damage rate parameters, in the case of glass plate of A=1,00m^2 with A_1=0,05 m^2 tensile-stressed by σ_1 = 20,0 Mpa, A_2=0,35 m^2 by σ_2 = 10,0 Mpa and A_3=0,60 m^2 by σ_3 = 5,0 Mpa, is possible to calculate by (4-2) the equidamaging uniform stress:

$$\bar{\sigma} = \left[\frac{1}{A_0} \left(\sigma_1^{9,25} \cdot A_1 + \sigma_2^{9,25} \cdot A_2 + \sigma_3^{9,25} \cdot A_3 \right) \right]^{\frac{1}{9,25}} \qquad \bar{\sigma} = 14,49 MPa$$

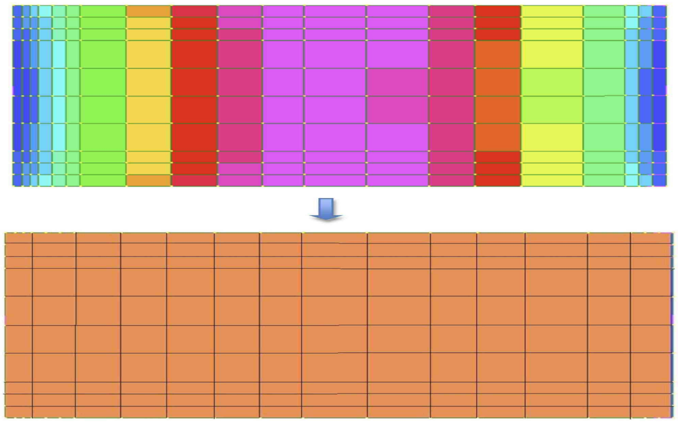

fig 5-1 the equidamaging uniform stress

Obviously a generic surface tensile stress field, obtained for instance by F.E.M. analysis, can be subdivided in tecnically uniformly tensile stressed areas (fig-5-1). Therefore it is possible to take into account the different safety level of two glass element with the same maximum tensile stress but with a completly different tensile stress field, unattainable with a deterministic allowable-stress approch. This safety verification process can lead to no-negligible economical consequences in terms of material amount used (thinner glass elements).

As can be seen, this simple calculation takes also into account the different failure stress between similar glass elements with different surface values A_0 and A_1. This phenomenon is called *size effect* and takes into account the obviously higher probability of a larger glass element to contain a deeper crack because of the random cracks distribution, by the following expression:

$$\alpha_{se} = (A_0 / A_1)^{1/9,95}$$

6 Conclusion

The present panorama of glass strength criteria that can be applied in the field of architectural glass structures is still far from being simple and homogeneous. Despite the wide variety of existing calculation models, none of them are preferred by glass designers or researchers since some methods are usable only for simple geometries and standard constraint set ups, while others can be applied to any element and service condition but are difficult to use due to their high calculation complexity. On the other hand, some other methods demonstrated to be totally inadequate because of their lack of sufficiently safe results.

With the new failure prediction method proposed in this paper we tried to develop a user-friendly tool while maintaining at the same time the precision of the most rigorous methods but avoiding their high calculation burden. This has been done with the aim of supporting structural engineers when tasked with designing glass structures. This aim has been pursued through the adoption of a reliable, easy to understand deterministic model, already used by designers, together with the simplified version of the more accurate glass strength model developed by Haldimann.

The new method, called *Design Crack Method*, is free from any empirical formulation. The crucial assessment parameter is a crack's depth, called *Design Crack*, which is independant on time and takes into account all the main factors that statistically govern the strength of glass during time, such as the static fatigue and the stress concentration at the apex of the surface flaws.

The great sensitivity of glass strength on the sequence and duration of external loading highlights the importance of defining significant stress time-history as standard loading conditions, a problem that could be solved within the frame of Eurocode activities.

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