New design concepts focused on reliability of structural glass elements

Marco Devigili[1,2], Maurizio Froli[1]
[1] Structural Engineering Department, University of Pisa
[2] Sant'Anna School of Advanced Studies, Pisa

Keywords

1= structural glass 2= reliability 3= fracture mechanics 4= stochastic process

Abstract

In spite of the growing use of glass in architecture, there has been no rigorous and complete method of structural glass designing so far.

Our research aims at a new and flexible approach to the problem. On the basis of structural reliability and fracture mechanics, a useful mathematical model of glass behaviour can be developed and applied to most cases, as for example to glass plates or glass beams subjected to various load histories.

Introduction

The difference between glass and other more commonly used materials consists in its fragility and this makes it difficult to develop analytic methods of structural dimensioning.

It is now generally accepted that glass failure does not occur when the theoretical tensile strength based on molecular forces is reached, but rather as a consequence of cracks due to production flaws that cannot be avoided, or to mechanical surface damage [2]. The cracks grow under tensile stresses until a critical depth at which the growth becomes unstable and this leads to the immediate breakage of the glass element.

The subcritical crack growth – also known as stress corrosion, or static fatigue – depends on the initial flaw distribution as well as on the entire stress history to which the flaw surrounding area has been subjected. It also depends on environmental conditions, above all humidity, that glass is exposed to.

The glass failure event can therefore be defined as extremely complex and aleatory.

The existing methods tend to bypass the aleatory property of glass failure by introducing equivalent quantities corrected by adequate coefficients. The main methods may be distinguished as follows:

- admissible stress methods, often adopted by producers or e.g. in [3];
- "American" methods: based on GFPM – Glass Failure Prediction Model [4,5,6];
- "European" methods [7,8,9,10,11].

The Lifetime Prediction Model (LPM) method, described in [12], in which the failure probability has been considered as depending on probability distribution of initial flaws, is an exception. The Crack Size Design described in [13] is also worth mentioning. In each point it presumes the existence of a design crack that equals a fractile of probability distribution of initial flaws for the whole pane.

No method found in the bibliography deals with the probabilistic property of load history, unless in order to define a stress fractile (the semi probabilistic method).

On the contrary, this paper aims at the direct involvement of the load history aleatority in the model by producing some simplifications. This method has been developed in [1].

Objectives

The present paper proposes a model which describes the reaching of the failure event at a particular point.

The definition of point in this paper can be understood as any pane point or any finite element deriving from pane discretization. In the following only the surface points will be referred to, because they use to produce major flaws and they are susceptible to mechanical damage and to higher stress (i.e. in case of flexural behaviour). The same method can be extended to internal points, provided that some parameters have been changed.

Irwin's proposal [14] consists in a verification based on so called stress intensity factor *K*: the brittle failure occurs when *K* becomes equal to K_{cr}, this is a material constant.

$$K(t)=Y\,\sigma(t)\sqrt{\pi\cdot a(t)}<K_{cr} \qquad (1)$$

where
σ is the nominal stress perpendicular to the crack plane;
a is the crack depth;
Y is a correction factor depending on the crack shape;
t is time.
The model is aimed at defining the glass element lifetime $T(X_0)$, that is the time it takes to reach K_{cr} for the first time, being $X_0 = \{\sigma(0), a(0)\}$ the initial state.

If *T* is a random variable, then the n-order moments are needed in order to determine *T*.

$$T_n(X_0, K_{cr})=\langle T^n\rangle=\int_{-\infty}^{+\infty}\tau^n\cdot p(\tau)\,d\tau \qquad (2)$$

given that
p(*x*) is the probability that *T* = x;
(*x*) is the expected value of x.
If the moments are known it will be easy to calculate the distribution mean and variance:

$$mean(T)=\langle T\rangle=T_1\,;\; var(T)=\langle (T-\langle T\rangle)^2\rangle=T_2-T_1^2\,. \qquad (3)$$

Hypotheses

1. The failure under tensile stresses occurs only when *K* equals K_{cr}, that is to say at the event of the unstable crack growth: theoretical tensile strength is considered infinite.
2. On the contrary, the failure under compressive stresses occurs when σ equals σ_c.
3. The probability to reach failure in the infinite time equals 1.
4. Initially there is a crack in every point; the crack depth a_0 follows the

given probability distribution.

5. The mechanical crack behaviour follows the elastic fracture mechanics with the *mode I propagation*.
   The empiric equation from [15] in particular is considered valid

$$\frac{da}{dt}=S\cdot K^{N}\geqslant 0 \tag{4}$$

   where *S,N* are parameters depending on environmental conditions, i.e. mainly atmospheric humidity. Their probability distributions are important to this model. Below they are considered as both known and constant, as they are, for example, in an indoor setting. This equation is true for $K>K_{th}$, where is a threshold value of the stress intensity factor. In [12] it is shown that considering
   $K_{th}\simeq 0$ is not only a safe assumption for design purposes but it is also practically irrelevant. The phenomenon of crack healing can also be neglected.
6. The crack depth always remains infinitesimal compared to glass pane thickness, the crack shape does not affect the phenomenon. Therefore $Y\simeq 1.12$ [15].
7. The major principal stress at a point is always perpendicular to the crack plane (safe assumption [12]).
8. The stress and the crack depth are random variables and their evolution depends only on their present values and not on their history *(Markov property)*.
9. The relation between these variables at two different times t_1 and t_2 does not depend on t_1 and t_2 but rather on $t_2 - t_1$.

Markov Process

The phenomenon is analysed as the stochastic Markov process, i.e. a time succession of random variables, the distribution of which is fully known if the state of the variable at one and sole preceding instant is also known. This can be expressed as follows:

$$p(x(t),t|x(\tau),\tau,\forall\tau<t)=p(x(t),t|x(\bar{t}),\bar{t},\bar{t}<t)\ .$$

In order to model glass behaviour Markov process with two variables has been introduced:
1. the nominal stress σ*(t)* occurring normally at the crack;
2. crack depth *α*(t).
The state variable is therefore $\boldsymbol{X}(t)=\{\sigma(t),a(t)\}$.
It is logical to hypothesize that there is no memory for *α* if the crack propagation speed is not excessive. On the contrary, the stress does not always show Markovian behaviour. Consider a component subject to a moving load; we can easily presume that the stresses in a temporal dimension are closely interrelated. Nevertheless, interesting results can be achieved with a Markovian stress history as well. For further details see [1]
The process is fully defined if its initial value $\boldsymbol{X}(0)=\{\sigma(0),a(0)\}$ and the increment function $\Xi(dt,\boldsymbol{X}(t),t)=\{d\sigma(t),da(t)\}$ have been defined.
It can be demonstrated [16] that

$$\Xi(dt,\boldsymbol{X}(t),t)=\{S(\boldsymbol{X}(t),t)dt,A(\boldsymbol{X}(t),t)dt\} \tag{5}$$

given that $S(\boldsymbol{X}(t),t),A(\boldsymbol{X}(t),t)$ are normally distributed with *M,V* respectively mean and covariance matrices.

$$\boldsymbol{M}(\boldsymbol{X},t)=\begin{Bmatrix}\beta_{\sigma}(\boldsymbol{X},t)\\ \beta_{a}(\boldsymbol{X},t)\end{Bmatrix}\qquad \boldsymbol{V}(\boldsymbol{X},t)=\begin{Bmatrix}\alpha_{\sigma}(\boldsymbol{X},t) & \alpha_{\sigma a}(\boldsymbol{X},t)\\ \alpha_{\sigma a}(\boldsymbol{X},t) & \alpha_{a}(\boldsymbol{X},t)\end{Bmatrix} \tag{6}$$

While the distribution regarding *dσ/dt* is closely dependent on the examined case, *dα/dt* can be defined after dsigma has been determined. Indeed:

$$\frac{da(\boldsymbol{X})}{dt}=\delta\cdot S\cdot K^{N}=\delta\, S\cdot Y^{N}\pi^{N/2}(\sigma+d\sigma)^{N}\sqrt{a^{N}} \tag{7}$$

$$\beta_{a}(\boldsymbol{X})=\left\langle\delta\cdot S\cdot Y^{N}\pi^{N/2}(\sigma+d\sigma)^{N}\sqrt{a^{N}}\right\rangle\overset{d\sigma\ll\sigma}{\simeq}\delta\cdot S\cdot Y^{N}\pi^{N/2}\sigma^{N}\sqrt{a^{N}} \tag{8}$$

$$\alpha_{a}(\boldsymbol{X})=\left\langle\left(\delta S\, Y^{N}\pi^{N/2}(\sigma+d\sigma)^{N}\sqrt{a^{N}}\right)^{2}\right\rangle-\beta_{a}^{2}(\boldsymbol{X})\overset{d\sigma\ll\sigma}{\simeq}0 \tag{9}$$

where $\delta=\begin{cases}1 & for\ K>K_{th}\\ 0 & for\ K\leqslant K_{th}\end{cases}$.

However, as has already been mentioned, we can consider $K_{th}\simeq 0$ without altering the results too much. The approximation simplifies the calculation because it is only a σ condition and not a condition between σ and *α*, but the approximation is not strictly necessary.

The hypothesis 9 implies that the process is stationary [16].

Life-expectation assessment depending on the initial conditions

As a consequence of random processes, the lifetime T is represented by a probability distribution.

$$p(T>t)=G(\boldsymbol{X}_0,t)=\int_{-\infty}^{K_{cr}}\int_{0}^{\infty}p\left(\left\{\frac{K}{Y\sqrt{\pi a}},a\right\},t\left|\left\{\frac{K_0}{Y\sqrt{\pi a_0}},a_0\right\},t_0\right.\right)da\,dK \tag{10}$$

From the backward Fokker-Planck equation extended to the bidimensional case [16]

$$-\frac{\partial}{\partial t_0}p(X(t)|X_0,t_0)=\frac{1}{2}\left[\alpha_\sigma(X_0)\frac{\partial^2}{\partial \sigma_0^2}p(X(t)|X_0,t_0)+2\alpha_{\sigma a}(X_0)\frac{\partial^2}{\partial \sigma_0 \partial a_0}p(X(t)|X_0,t_0)+ +\alpha_a(X_0)\frac{\partial^2}{\partial a_0^2}p(X(t)|X_0,t_0)\right]+\beta_\sigma\frac{\partial}{\partial \sigma_0}p(X(t)|X_0,t_0)+\beta_a\frac{\partial}{\partial a_0}p(X(t)|X_0,t_0) \quad (11)$$

considering that in case of a stationary process it occurs that

$$-\frac{\partial}{\partial t_0}p(X(t)|X_0,t_0)=\frac{\partial}{\partial t}p(X(t)|X_0,t_0) \quad (12)$$

by integrating in *dα dK* similarily to the equation (10) the following result is obtained:

$$\frac{\partial}{\partial t}G(X_0,t)=\frac{1}{2}\left[\alpha_\sigma(X_0)\frac{\partial^2}{\partial \sigma_0^2}G(X_0,t)+2\alpha_{\sigma a}(X_0)\frac{\partial^2}{\partial \sigma_0 \partial a_0}G(X_0,t)+ +\alpha_a(X_0)\frac{\partial^2}{\partial a_0^2}G(X_0,t)\right]+\beta_\sigma\frac{\partial}{\partial \sigma_0}G(X_0,t)+\beta_a\frac{\partial}{\partial a_0}G(X_0,t) \quad (13)$$

From (10) it can be infered that 1-*G* is the cumulative distribution function of *T*, thus the *T* probability density function is

$$p(T=t)=\frac{\partial}{\partial t}[1-G(X_0,t)]=-\frac{\partial}{\partial t}G(X_0,t) \quad (14)$$

And the moments of *T*, given by the equation (2), become

$$T_n(X_0)=\int_0^{+\infty} t^n\left[-\frac{\partial}{\partial t}G(X_0,t)\right]dt \quad (15)$$

By integrating by parts the following is obtained

$$T_n(X_0)=n\int_0^{+\infty} t^{n-1}G(X_0,t)\,dt \quad (16)$$

Now we can multiply by t^{n-1} and integrate in time the (13), and substitute the equation (15) in the left hand side term and the equation (16) in the right hand side term. The result will be a differential equation system depending on the T_n moments:

$$-nT_{n-1}(X_0)=\frac{1}{2}\left[\alpha_\sigma(X_0)\frac{\partial^2}{\partial \sigma_0^2}T_n(X_0)+2\alpha_{\sigma a}(X_0)\frac{\partial^2}{\partial \sigma_0 \partial a_0}T_n(X_0)+\alpha_a(X_0)\frac{\partial^2}{\partial a_0^2}T_n(X_0)\right]+ +\beta_\sigma\frac{\partial}{\partial \sigma_0}T_n(X_0)+\beta_a\frac{\partial}{\partial a_0}T_n(X_0) \quad (17)$$

with boundary conditions

$$T_n\left(\frac{K_{cr}}{Y\sqrt{\pi a_0}},a_0\right)=0 \qquad T_n(\sigma_c,a_0)=0 \quad (18)$$

expression of the cases in which the initial condition immediately leads to failure.
The following boundary condition can be added in order to solve the differential equations:

$$T_n(\sigma_0,a_{max})=0 \quad (19)$$

given that α_{max} is a large crack depth, which is very unlikely reached.

General equations can be simplified by using $d\sigma \ll \sigma$. In this case can be assumed and *dα*, once *X* has been fixed, becomes deterministic. Thus $\alpha_a=\alpha_{\sigma a}=0$.

By means of these equations the T_n can be determined and consequently the mean and the variance of *T*. If the order moments superior to the second order are neglected, the mean and the variance give us a complete characterization of the *T* probability distribution depending on X_0.

Examination of initial flaw distribution

Note that the initial flaw distribution has not been treated so far, as the issue has been uncoupled from the stress history. Hence let us consider the *T* distribution dependent on (σ_0,α_0); the introduction of a probability law α_0 for should be sufficient to obtain a *T* distribution depending on σ_0 alone.

$$\bar{T}_n(\sigma_0)=\int_{a_{min}}^{a_{max}} T_n(X_0)\cdot p(a_0)\,da_0 \quad (20)$$

The most frequently adopted distributions in research related to this subject for α_0 are the following [8,10,12,13]:
- Pareto distribution (punctual) [12];
- Weibull distribution of extreme values (for the pertinence area of a finite element).

In our model we are not considering the crack extending due to mechanical damage, but it can be introduced through stresses (forecasting temporally limited peaks) or through the introduction of a “worst case” scenario for initial flaws.

Lifetime distribution

If σ_0 is taken for granted, the T probability distribution function $p(T=t)$ can be reached. This function expresses directly the local reliability. In case that the function is normally distributed it can be seen that:

$$p(T=t)=N(T_1(\sigma_0),T_2(\sigma_0)-T_1^2(\sigma_0)) \tag{21}$$

given that $N(x,y)$ is the normal random variable with the mean x and the variance y.

It is possible to search for the real T distribution by calculating the order moments superior to the second order and by solving other differential equations, which are analogous to (17).

The final verification is carried out on the probabilistic level by confronting two probabilities:

$$p(T<t_d)\leqslant p_{fd} \tag{22}$$

where
t_d is the required design lifetime;
p_{fd} is the maximum allowed failure probability.
p_{fd} depends on the structural importance of the detail and on the existence of post-failure resources.

Extension from Point to Whole Pane or Beam

There are numerous modes of extending the method from point to whole glass pane or beam:

- Subdivision in finite elements: Points become nodes in the FE model. Each node is associated to distribution of maximum initial flaws present in the pertinence area and stresses deriving from analysis. The survival probability is given from the product of all the node survival probabilities, after the joint survival probability has been deducted.
- Subdivision in comparable stress zones: Panes or beams are divided into zones with a reasonably uniform stress distribution. For each zone, the maximum stress and the probability distribution of maximum initial flaws on the relevant area should be considered. The advantage of this method is that it can be applied in case of zones which bear much higher stresses than others and/or which present larger cracks (for example beam edges). The analysis could be restricted to these zones only, and the survival probability is given from the product of all the zone survival probabilities, after the joint survival probability has been deducted.
- *Crack Size Design*: The most immediate, but perhaps overly safety-oriented procedure involves considering the major initial flaw and the major stress for the whole component. The verification is carried out as a point verification.

Examples

- **Liouville processes**. Liouville processes are deterministic, that is to say with the nil variance. The equations (17) become

$$-1=\beta_\sigma(\mathbf{X_0})\frac{\partial}{\partial\sigma_0}T(\mathbf{X_0})+\beta_a(\mathbf{X_0})\frac{\partial}{\partial a_0}T(\mathbf{X_0}) \tag{23}$$

Constant stress $\sigma(t)=\sigma_0>0$

The obtained equation is identical to the already known equation of classical stress corrosion theory [2]:

$$\begin{cases} SY^N\pi^{N/2}\sigma_0^N a_0^{N/2}\dfrac{\partial}{\partial a_0}T(\mathbf{X_0})=-1 \\ T(\sigma_{0,}a_{cr})=0 \end{cases} \Rightarrow T=\frac{2}{N-2}\frac{a_0^{-(N-2)/2}-a_{cr}^{-(N-2)/2}}{SY^N\pi^{N/2}\sigma^N} \tag{24}$$

where $a_{cr}=(K_{cr}/(Y\sqrt{\pi}\sigma_0))^2$ is the critical crack size.

Uniformly growing stress: $\dfrac{d\sigma(t)}{dt}=s>0$ and constant

$$\begin{cases} \beta_\sigma\dfrac{\partial}{\partial\sigma_0}T(\mathbf{X_0})+\delta SY^N\pi^{N/2}\sigma_0^N a_0^{N/2}\dfrac{\partial}{\partial a_0}T(\mathbf{X_0})=-1 \\ T\left(\sigma_{0,}(K_{cr}/(Y\sqrt{\pi}\sigma_0))^2\right)=0 \end{cases} \tag{25}$$

- **Real stochastic processes**
Almost uniformly growing stress: $\dfrac{d\sigma(t)}{dt}=\beta_\sigma+W(0,\alpha_\sigma)$

$W(0,\alpha_\sigma)$ is a white noise with the nil mean and a constant α_σ variance, β_σ is a constant.

$$\begin{cases} \dfrac{1}{2}\alpha_\sigma\dfrac{\partial^2}{\partial\sigma_0^2}T_n(\mathbf{X_0})+\beta_\sigma\dfrac{\partial}{\partial\sigma_0}T_n(\mathbf{X_0})+\delta SY^N\pi^{N/2}\sigma_0^N a_0^{N/2}\dfrac{\partial}{\partial a_0}T_n(\mathbf{X_0})=-nT_{n-1}(\mathbf{X_0}) \\ T_n\left(\sigma_{0,}(K_{cr}/(Y\sqrt{\pi}\sigma_0))^2\right)=0 \;\wedge\; T_n(\sigma_c,a_0)=0 \;\wedge\; T_n(\sigma,a_{max})=0 \end{cases} \tag{26}$$

Summary and Conclusions

This paper aims to propose a new approach to the reliability study of structural glass elements.

After having assumed specific hypotheses it is put forward that it is possible to model a random propagation of glass cracks as Markov stochastic process. The advantage of using this model is the possibility to obtain probability distributions of survival time of a glass element depending on probability law related to stress evolution during time, as well as the probability distribution of initial flaws.

This method can be extended to other cases but in these cases different equations are obtained. See [1] for more examples.

Knowing the probability distributions allows a verification to be elaborated on the probability level, i.e. a comparison between available reliability and target reliability depending on the structural importance of the detail.

The topic seems to be promising but it still requires an intensive research, which the authors are committing to at the moment, in order for the model to be fully developed.

Acknowledgements

The authors would like to thank prof. Marco Beghini, ing. Giovanni Buratti, Marco Degiorgi, prof. Franco Flandoli, dr. Paolo D. Frumento, prof. Massimo Gobbino. We are also grateful to the Department of Structural Engineering of the University of Pisa.

Last but not least we would like to express our appreciation to Prof. Dubravka Vrsaljko Devigili and Prof. Maria Mcquillan for their translation consultancy.

References

[1] DEVIGILI, M., FROLI, M. (supervisor), Dimensionamento e affidabilità delle strutture in vetro, MSc Thesis, Università di Pisa, 2006.
[2] MENČÍK, J., Strength and Fracture of Glass and Ceramics, Elsevier, 1992.
[3] Technische Regeln für die Verwendung von linienförmig gelagerten Verglasungen, DIBt, 1998. Technische Regeln für die Verwendung von absturzsichernden Verglasungen, DIBt, 2003.
[4] BEASON, W.L. AND MORGAN, J.R., Glass Failure Prediction Model, Journal of Structural Engineering (ASCE), 110(2), 197-212, 1984.
[5] CAN/CGSB 12.20-M89, Structural Design of Glass for Buildings, 1989.
[6] ASTM E 1300-04, Standard Practice for Determining the Minimum Thickness and Type of Glass Required to Resist a Specific Load, 2004.
[7] SHEN, X., Entwicklung eines Bemessungs- und Sicherheitskonzeptes für den Glasbau, PhD Thesis, TU Darmstadt, 1997.
[8] GÜSGEN, G., Bemessung tragender Bauteile aus Glas, PhD Thesis, RWTH Aachen, 1998.
[9] SEDLACEK, G., et al., Glas im Konstruktiven Ingenieurbau, Ernst & Sohn, 1999.
[10] SIEBERT, G., Entwurf und Bemessung von tragenden Bauteilen aus Glas, Ernst & Sohn, 2001.
[11] prEN 13474, Glass in Buildings - Design of Glass Panes, 1999-2000.
[12] HALDIMANN, M., Fracture Strength of Structural Glass Elements - Analytical and Numerical Modelling, Testing and Design, PhD Thesis, EPFL, 2006.
[13] PORTER, M., Aspects of Structural Design with Glass, PhD Thesis, Oxford University, 2001.
[14] IRWIN, G., Analysis of Stresses and Strains near the End of a Crack Traversing a Plate, Journal of Applied Mechanics, 29(4), 651-654, 1957
[15] EVANS, A.G. and WIEDERHORN, S.M. Proof Testing of Ceramic Materials - An Analytical Basis for Failure Prediction, International Journal of Fracture, 10(3), 379-392, 1974. Reprinted in 26(4), 355-368, 1984.
[16] GARDINER, C. W., Handbook of Stochastic Methods, Springer, 1997.
[17] GILLESPIE, D. T., Markov Processes, Academic Press, 1992.